ISRAEL JOURNAL OF MATHEMATICS 90 (1995), 423–428

ORTHOGONAL SUMS OF SEMIGROUPS*

BY

STOJAN BOGDANOVIĆ AND MIROSLAV ĆIRIĆ

Faculty of Economics, University of Niš, 18000 Niš, Yugoslavia

ABSTRACT

The purpose of this paper is to prove that every semigroup with the zero is an orthogonal sum of orthogonal indecomposable semigroups. We prove that the set of all 0-consistent ideals of an arbitrary semigroup with the zero forms a complete atomic Boolean algebra whose atoms are summands in the greatest orthogonal decomposition of this semigroup.

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers and $S = S^0$ means that S is a semigroup with the zero 0. If $S = S^0$, we will write 0 instead $\{0\}$ and, if A is a subset of S, we will write $A^* = A - 0$, $A^0 = A \cup 0$, $A' = (S - A)^0$. For an element a of a semigroup S, $J(a)$ will denote the principal ideal of S generated by a.

A lattice L is **complete** if every nonempty subset of L has a least upper bound and a greatest lower bound. An element a of a lattice L with the zero 0 is an **atom** of L if $a > 0$ and there exists no $x \in L$ such that $a > x > 0$. A complete Boolean algebra B is **atomic** if every element of B is the least upper bound of some set of atoms of B. If L is a distributive lattice with the zero and the unity, then the set of all elements of L having a complement in L is a Boolean algebra which we call the **greatest Boolean subalgebra** of L. By $\mathcal{I}d(S)$ we denote the lattice of ideals of a semigroup S. For $S = S^0$, this is a distributive lattice with the zero 0 and the unity S.

A semigroup $S = S^0$ is an **orthogonal sum** of semigroups $S_\alpha, \alpha \in Y$, in notation $S = \Sigma_{\alpha \in Y} S_\alpha$, if $S_\alpha \neq 0$, for all $\alpha \in Y$, $S = \bigcup_{\alpha \in Y} S_\alpha$ and $S_\alpha \cap S_\beta = S_\alpha S_\beta = S_\beta S_\alpha = 0$, for all $\alpha, \beta \in Y, \alpha \neq \beta$. In this case, the family $\mathcal{D} =$

---

* Supported by Grant 0401A of RFNS through Math. Inst. SANU.
Received February 18, 1993 and in revised form January 9, 1994

$\{S_\alpha \mid \alpha \in Y\}$ is an **orthogonal decomposition** of S and S_α are **orthogonal summands** of S or **summands** in $\mathcal{D}$. If $\mathcal{D}$ and $\mathcal{D}'$ are two orthogonal decompositions of a semigroup $S = S^0$, then we say that $\mathcal{D}$ is **greater than** $\mathcal{D}'$ if each member of $\mathcal{D}$ is a subset of some member of $\mathcal{D}'$. A semigroup $S = S^0$ is **orthogonal indecomposable** if $\mathcal{D} = \{S\}$ is the unique orthogonal decomposition of S.

For undefined notions and notations we refer to [3], [4] and [10].

Orthogonal decompositions of semigroups were first studied by E.S.Lyapin, [7], [8], and by Š. Schwarz, [14]. Orthogonal sums of 0-simple semigroups and of null semigroups, and some special types of these, are considered by Š. Schwarz, [14], and by A. H. Clifford and G. B. Preston, [4]. Various characterizations of orthogonal sums of completely 0-simple semigroups (i.e. of primitive regular semigroups) are given by P. S. Venkatesan, [12], [13], O. Steinfeld, [11], G. Lallement and M. Petrich, [6], G. B. Preston, [9], and T. E. Hall, [5]. G. Lallement and M. Petrich, [6], also described orthogonal sums of semigroups having a prime zero ideal.

The purpose of this paper is to prove that there exists a **greatest orthogonal decomposition** of an arbitrary semigroup $S = S^0$ and to describe it. We consider 0-consistent ideals of an arbitrary semigroup $S = S^0$, we prove that they form an atomic Boolean algebra whose atoms form the greatest orthogonal decomposition of S, and also, we prove that every complete atomic Boolean algebra is isomorphic to the Boolean algebra of 0-consistent ideals of some semigroup with zero.

A subset A of a semigroup S is **consistent** if for $x, y \in S$, $xy \in A$ implies $x, y \in A$. A subset A of a semigroup $S = S^0$ is **0-consistent** if A^* is consistent. We will describe a role of 0-consistent ideals in orthogonal decompositions of semigroups with zero.

It is obvious that a subset of a semigroup S is consistent if and only if its complement in S is an ideal. By this it follows that:

LEMMA 1: *The following conditions for an ideal A of a semigroup $S = S^0$ are equivalent.*

(i) *A is 0-consistent;*

(ii) *A' is an ideal of S;*

(iii) *A is an orthogonal summand of S.*

It is easy to prove the following three lemmas:

LEMMA 2: *Let A be a 0-consistent ideal of a semigroup $S = S^0$ and let B be a 0-consistent ideal of A. Then B is a 0-consistent ideal of S.*

LEMMA 3: *Let $A_i, i \in I$, be a family of 0-consistent ideals of a semigroup $S = S^0$. Then $\bigcap_{i\in I} A_i$ and $\bigcup_{i\in I} A_i$ are 0-consistent ideals of S.*

A 0-consistent ideal A of a semigroup $S = S^0$ is a **proper 0-consistent ideal** of S if $A \neq 0$ and $A \neq S$.

LEMMA 4: *A semigroup $S = S^0$ is orthogonal indecomposable if and only if S has no proper 0-consistent ideals.*

Let $S = S^0$. For $a \in S$, by $\Delta(a)$ we will denote the intersection of all 0-consistent ideals of S containing a. By Lemma 3, $\Delta(a)$ is **the smallest 0-consistent ideal of S containing a**, and we will call it **principal 0-consistent ideal of S generated by a**.

Let we introduce a relation δ on S by

$$a\delta b \quad \leftrightarrow \quad \Delta(a) = \Delta(b) \qquad (a, b \in S).$$

Then δ is an equivalence on S, and by Δ_a we will denote the δ-class of S containing an element $a \in S$. It is clear that $\Delta_0 = \Delta(0) = 0$.

LEMMA 5: *If $S = S^0$ and $a, b \in S$, then*

$$ab \neq 0 \Longrightarrow \Delta(ab) = \Delta(a) = \Delta(b).$$

The principal 0-consistent ideals may be described also by another way. Let τ be a relation on a semigroup $S = S^0$, defined by

$$x\,\tau\,y \leftrightarrow J(x) \cap J(y) \neq 0, \quad \text{for } x, y \in S^*, \qquad 0\,\tau\,0.$$

Clearly, τ is reflexive and symmetric. Let $\overline{\tau}$ be the transitive closure of τ, and let $T(a) = \{x \in S \mid x\,\overline{\tau}\,a\} \cup 0$, for $a \in S$.

LEMMA 6: *Let $S = S^0$, $a \in S$. Then $\Delta(a) = T(a)$.*

Proof: If $x, y \in S$ satisfy $xy \in (T(a))^*$, then $xy\,\overline{\tau}\,a$; since $xy \in J(xy) \cap J(x) \cap J(y)$, then $x\,\tau\,xy$ and $y\,\tau\,xy$. Therefore $x\,\overline{\tau}\,a$ and $y\,\overline{\tau}\,a$, i.e. $x, y \in (T(a))^*$. Thus, $T(a)$ is a 0-consistent subset of S.

If for $b \in S$, $ab \neq 0$, then $a \,\tau\, ab$ (because $ab \in J(ab) \cap J(a)$), and therefore $ab \in T(a)$. The same holds for ba, and thus, $T(a)$ is an ideal of S.

We obtained that $T(a)$ is a 0-consistent ideal of S, containing a. Therefore, $T(a) \supseteq \Delta(a)$ by definition of $\Delta(a)$.

Conversely, if $x \in \Delta(a)$ and $0 \neq y \in S$ satisfy $y \,\tau\, x$, i.e. $pyq = uxv \neq 0$ for some $u, v, p, q \in S^1$, then $y \in (\Delta(a))^*$, because $pyq = uxv \in (\Delta(a))^*$ and $\Delta(a)$ is 0-consistent. Therefore, $T(a) \subseteq \Delta(a)$, which proves the lemma. ∎

LEMMA 7: *Let $a \neq 0$ be an element of a semigroup $S = S^0$. Then*
(A1) *$\Delta(a)$ is orthogonal indecomposable;*
(A2) *$\Delta(a) = \Delta_a^0$.*

Proof: (A1) If $\Delta(a)$ has a proper 0-consistent ideal A, then A and $B = (\Delta(a) - A)^0$ are proper 0-consistent ideals of $\Delta(a)$ and S (by Lemmas 1 and 2), and $a \in A$ or $a \in B$, which contradicts our hypothesis that $\Delta(a)$ is the smallest 0-consistent ideal of S containing a. Thus, $\Delta(a)$ has no proper 0-consistent ideals, so by Lemma 4, $\Delta(a)$ is orthogonal indecomposable.

(A2) Let $\Delta_a^0 = \Delta_a \cup 0$. For $x \in \Delta_a$ we obtain that $\Delta(x) = \Delta(a)$, so $x \in \Delta(x) = \Delta(a)$, whence we obtain that $\Delta_a \subseteq \Delta(a)$, i.e. $\Delta_a^0 \subseteq \Delta(a)$.

Suppose that $\Delta_a^0 \neq \Delta(a)$. Then there exists $x \in \Delta(a)$ such that $x \neq 0$ and $\Delta(x) \neq \Delta(a)$. Hence, $\Delta(x)$ is a proper 0-consistent ideal of $\Delta(a)$, i.e. $\Delta(a)$ is not orthogonal indecomposable (Lemma 4), in contradiction to (A1). Therefore, $\Delta_a^0 = \Delta(a)$. ∎

THEOREM 1: *The set $\mathfrak{B}(S)$ of all 0-consistent ideals of a semigroup $S = S^0$ is a complete atomic Boolean algebra and it is the greatest Boolean subalgebra of $\mathcal{I}d(S)$.*

Furthermore, every complete atomic Boolean algebra is isomorphic to the Boolean algebra of 0-consistent ideals of some semigroup with zero.

Proof: By Lemma 3 we obtain that $\mathfrak{B}(S)$ is a sublattice of $\mathcal{I}d(S)$, and by Lemma 1 we obtain that it is the greatest Boolean subalgebra of $\mathcal{I}d(S)$. By Lemma 3 we have that $\mathfrak{B}(S)$ is complete. By Lemmas 7 (A1) and 4 we have that nonzero principal 0-consistent ideals of S are atoms in $\mathfrak{B}(S)$. Also, if A is a nonzero 0-consistent ideal of S, then $A = \bigcup\{\Delta(a) \mid a \in A, a \neq 0\}$, i.e. A is a union of atoms. Thus, $\mathfrak{B}(S)$ is an atomic Boolean algebra.

Further, let B be a complete atomic Boolean algebra and let Y be the set of atoms of B. To each $a \in Y$ let we associate an orthogonal indecomposable semigroup S_a (for example a 0-simple semigroup) such that $S_a \cap S_b = 0$, if $a \neq b$, where 0 is the common zero of $S_a, a \in Y$. Let $S = \Sigma_{a \in Y} S_a$. It is clear that $S_a, a \in Y$, are all nonzero principal 0-consistent ideals of S, i.e. all the atoms of $\mathfrak{B}(S)$. It is well known that every complete atomic Boolean algebra is isomorphic to the Boolean algebra of all subsets of its atoms. The sets of atoms of B and of $\mathfrak{B}(S)$ have the same cardinality, therefore B and $\mathfrak{B}(S)$ are isomorphic. ∎

THEOREM 2: *Every semigroup $S = S^0$ has a greatest orthogonal decomposition; its summands are all the atoms of $\mathfrak{B}(S)$.*

Proof: By Lemmas 5 and 7 we obtain that S is an orthogonal sum of its nonzero principal 0-consistent ideals, which are orthogonal indecomposable. ∎

COROLLARY 1: *The following conditions on a semigroup $S = S^0$ are equivalent:*

(i) *S is an orthogonal sum of 0-simple semigroups and of null semigroups;*
(ii) *every ideal of S is 0-consistent;*
(iii) $(\forall x, y \in S) xy \neq 0 \Rightarrow x, y \in SxyS$;
(iv) *$\mathcal{I}d(S)$ is a Boolean algebra.*

Proof: The equivalence (ii)↔(iv) follows by Theorems 1 and 2. The rest is proved by A. H. Clifford and G. B. Preston in [4] and by Š. Schwarz in [14]. ∎

ACKNOWLEDGEMENT: The authors are indebted to the referee for several useful comments and suggestions concerning the presentation of this paper.

References

[1] S. Bogdanović and M.Ćirić, *Primitive π-regular semigroups*, Japan Academy. Proceedings, Vol. 68, Series A **10** (1992), 334–337.

[2] M. Ćirić and S. Bogdanović, *Semilattice decompositions of semigroups*, to appear.

[3] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, I, American Mathematical Society, 1961.

[4] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, II, American Mathematical Society, 1967.

[5] T. Hall, *On the natural order of $\mathcal{J}$-class and of idempotents in a regular semigroup*, Glasgow Mathematical Journal **11** (1970), 167–168.

[6] G. Lallement and M. Petrich, *Décomposition I-matricielles d'une demi-groupe*, Journal de Mathématiques Pures et Appliquées **45** (1966), 67–117.

[7] E. S. Lyapin, *Normal complexes of associative systems*, Izvestiya Akademii Nauk SSSR **14** (1950), 179–192 (in Russian).

[8] E. S. Lyapin, *Semisimple commutative associative systems*, Izvestiya Akademii Nauk SSSR **14** (1950), 367–380 (in Russian).

[9] G. B. Preston, *Matrix representations of inverse semigroups*, Journal of the Australian Mathematical Society **9** (1969), 29–61.

[10] G. Szász, *Théorie des treillis*, Co-édition Akadémiai Kiadó, Budapest et Dunod, Paris, 1971.

[11] O. Steinfeld, *On semigroups which are unions of completely 0-simple semigroups*, Czechoslovak Mathematical Journal **16** (1966), 63–69.

[12] P. S. Venkatesan, *On a class of inverse semigroups*, American Journal of Mathematics **84** (1962), 578–582.

[13] P. S. Venkatesan, *On decomposition of semigroups with zero*, Mathematische Zeitschrift **92** (1966), 164–174.

[14] Š. Schwarz, *On semigroups having a kernel*, Czechoslovak Mathematical Journal **1 (76)** (1951), 259–301 (in Russian).